Physics 606 Final Exam Solution

1. (a) $\frac{d}{dr}\left(4\pi r^2 \cdot \frac{1}{4\pi}\frac{4}{a_0^3} e^{-2r/a_0}\right) = 0$

$$\Rightarrow 0 = \frac{d}{dr}\left(r^2 e^{-2r/a_0}\right) = \left[2r + r^2\left(-\frac{2}{a_0}\right)\right]e^{-2r/a_0} \Rightarrow \boxed{r = a_0}$$

radius of orbit in Bohr model

(b) $\boxed{\langle r^2 \rangle =} \frac{1}{4\pi}\frac{4}{a_0^3}\int_0^\infty dr\, 4\pi r^2\, e^{-r/a_0}\, r^2\, e^{-r/a_0}$ $\quad u \equiv \frac{2r}{a_0}$

$$= \frac{4}{a_0^3}\int_0^\infty dr\, r^4 e^{-2r/a_0} = \frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^5 \underbrace{\int_0^\infty du\, u^4 e^{-u}}_{= \Gamma(5) = 4! = 24}$$

$$= \frac{4}{32}\cdot 24\, a_0^2 = \boxed{3a_0^2}$$

$$\Rightarrow \boxed{\langle x^2 \rangle =} \frac{1}{3}\langle (x^2+y^2+z^2)\rangle = \frac{1}{3}\langle r^2\rangle = \boxed{a_0^2}$$

(c) one way, since p_r is Hermitian:

$$\boxed{\langle p_r^2 \rangle =} \frac{1}{4\pi}\frac{4}{a_0^3}\int_0^\infty dr\cdot 4\pi r^2\left[-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)e^{-\frac{r}{a_0}}\right]^*\left[-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)e^{-\frac{r}{a_0}}\right]$$

$$= \frac{4\hbar^2}{a_0^3}\int_0^\infty dr\, r^2\left(-\frac{1}{a_0}+\frac{1}{r}\right)^2 e^{-2r/a_0}$$

$$= \frac{4\hbar^2}{a_0^3}\left[\frac{1}{a_0^2}\int_0^\infty dr\, r^2 e^{-2r/a_0} - \frac{2}{a_0}\int_0^\infty dr\, r\, e^{-2r/a_0} + \int_0^\infty dr\, e^{-2r/a_0}\right]$$

$$= \frac{4\hbar^2}{a_0^3}\left[\frac{1}{a_0^2}\left(\frac{a_0}{2}\right)^3 \underbrace{\Gamma(3)}_{=2!=2} - \frac{2}{a_0}\left(\frac{a_0}{2}\right)^2\underbrace{\Gamma(2)}_{=1!=1} + \frac{a_0}{2}\underbrace{\Gamma(1)}_{=0!=1}\right]$$

$$= \frac{4\hbar^2}{a_0^3}\cdot a_0\left(\frac{1}{4}-\frac{1}{2}+\frac{1}{2}\right)$$

$$= \boxed{\frac{\hbar^2}{a_0^2}}$$

$$\boxed{\langle p_x^2\rangle =} \frac{1}{3}\langle p_r^2\rangle = \boxed{\frac{1}{3}\frac{\hbar^2}{a_0^2}}$$

Then, since $\langle p_x\rangle = 0$ & $\langle x\rangle = 0$, $\boxed{\Delta p_x = \frac{1}{\sqrt{3}}\frac{\hbar}{a_0}}$ & $\boxed{\Delta x = a_0}$

so $\boxed{\Delta p_x \Delta x =} \frac{1}{\sqrt{3}}\hbar \boxed{>} \frac{1}{\sqrt{4}}\hbar = \boxed{\frac{1}{2}\hbar}$.

2. $\boxed{\Delta E_{1s} =} \langle 1s | \Delta V | 1s \rangle$

$$= \frac{4}{4\pi a_0^3} \int_0^R dr \cdot 4\pi r^2 e^{-2r/a_0} (-ke^2) \left(\frac{1}{R} - \frac{1}{r}\right)$$

$$= \frac{4ke^2}{a_0^3} \left[\int_0^R dr\, r \underbrace{e^{-2r/a_0}}_{\approx 1} - \frac{1}{R} \int_0^R dr\, r^2 \underbrace{e^{-2r/a_0}}_{\approx 1} \right]$$

$$\approx \frac{4ke^2}{a_0^3} \left[\left[\frac{r^2}{2}\right]_0^R - \frac{1}{R} \left[\frac{r^3}{3}\right]_0^R \right]$$

$$= \frac{4ke^2}{a_0^3} R^2 \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{= 1/6}$$

$$= \boxed{\frac{2}{3} \left(\frac{R}{a_0}\right)^2 \frac{ke^2}{a_0}}$$

$$\boxed{= \frac{4}{3} \times 10^{-10} \frac{ke^2}{2a_0}}$$

$$\boxed{\sim 10^{-7}\ \text{eV}}$$

$$3.\ (a)\ f(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\underbrace{\int_0^\infty dr\, \cancel{r}\, \frac{1}{2i}\left(e^{iqr} - e^{-iqr}\right)\cdot a\,\frac{e^{-Kr}}{\cancel{r}}}_{=\frac{a}{2i}I}$$

$$I = \int_0^\infty dr\, e^{-Kr}\left(e^{iqr} - e^{-iqr}\right)$$

$$= \left[\frac{e^{-(K-iq)r}}{-(K-iq)}\right]_0^\infty - \left[\frac{e^{-(K+iq)r}}{-(K+iq)}\right]_0^\infty$$

$$= \frac{1}{K-iq} - \frac{1}{K+iq} = \frac{(K+iq)-(K-iq)}{(K-iq)(K+iq)} = \frac{2iq}{K^2+q^2}$$

Then $\boxed{f(\theta) =} -\frac{2m}{\hbar^2}\frac{1}{q}\frac{a}{2i}\frac{2iq}{K^2+q^2}$

$$= \boxed{-\frac{a}{4E_{\hbar}\sin^2\left(\frac{\theta}{2}\right) + \frac{\hbar^2K^2}{2m}}}$$

since $E_{\hbar} = \frac{\hbar^2 k^2}{2m}$ and $q = 2k\sin\left(\frac{\theta}{2}\right)$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2}$$

(b) same as Rutherford when $K \to 0$, $a \to ZZ'e^2$

(c) As $q \to 0$,

$$\frac{d\sigma}{d\Omega} \to \left(\frac{a}{\frac{\hbar^2K^2}{2m}}\right)^2 .$$

4. (a) $|\psi(t)\rangle$ satisfies differential equation

(b). If $|\psi(t_0)\rangle = |i\rangle$, $\langle n|i\rangle = 0$ and

$$\langle n|\psi(t)\rangle = \frac{1}{\hbar}\int_{t_0}^{t} dt' \langle n| e^{iH_0 t'/\hbar} V_{t'} e^{-iH_0 t'/\hbar} |i\rangle$$

$$= \frac{1}{\hbar}\int_{t_0}^{t} dt' \, e^{i(\mathcal{E}_n - \mathcal{E}_i)t'/\hbar} \langle n|V_{t'}|i\rangle$$

with $P_{i\to n} = |\langle n|\psi(t)\rangle|^2$

(c) $\boxed{V_t =} - (-e\,\mathcal{E}_0 e^{-t/\tau}) z + \text{constant}$, but $\langle n|\text{constant}|i\rangle$
$= \text{constant}\langle n|i\rangle = 0$
so take constant $= 0$

$$= \boxed{e\,\mathcal{E}_0 e^{-t/\tau} r\cos\theta}$$

(d) $$\langle n|z|i\rangle = \int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{2}{a_0^{3/2}} e^{-r/a_0} \frac{1}{\sqrt{4\pi}} (r\cos\theta)$$
$$\times \frac{1}{(2a_0)^{3/2}} \frac{r}{a_0\sqrt{3}} e^{-r/2a_0} \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$\int_0^\pi d\theta \cos^2\theta \sin\theta = -\int_1^{-1} \mu^2 d\mu \quad , \mu \equiv \cos\theta$$
$$= +\int_{-1}^{1} \mu^2 d\mu = \left[\frac{\mu^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

so $$\langle n|z|i\rangle = 2\pi \frac{2}{a_0^{3/2}} \frac{1}{\sqrt{4\pi}} \frac{1}{(2a_0)^{3/2}} \frac{1}{a_0\sqrt{3}} \sqrt{\frac{3}{4\pi}} \cdot \frac{2}{3}$$
$$\times \underbrace{\int_0^\infty dr\, r^4 e^{-3r/2a_0}}_{\equiv J}$$

with $$J = \left(\frac{2a_0}{3}\right)^5 \underbrace{\int_0^\infty du\, u^4 e^{-u}}_{= \Gamma(5) = 4! = 24}$$

Then $$\boxed{\langle n|z|i\rangle = \frac{2^7\sqrt{2}}{3^5} a_0}$$ [after collecting powers of 2, 3, a_0]

(e) After doing the integral, one obtains

$$P(1s\to 2p) = \frac{2^{15} a_0^2 e^2 \mathcal{E}_0^2}{3^{10}\hbar^2\left(\frac{1}{\tau^2}+\omega^2\right)} \left(e^{-\frac{2}{\tau}t} - e^{-\frac{t}{\tau}} \cdot 2\cos(\omega t) + 1\right).$$

(f) As $t\to\infty$,

$$\boxed{P(1s\to 2p) \to \frac{2^{15} a_0^2 e^2 \mathcal{E}_0^2}{3^{10}\hbar^2\left(\frac{1}{\tau^2}+\omega^2\right)}}$$